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DIGITAL MEASUREMENT OF DIFFERENTIAL TIME DELAY OF PSEUDO RANDOM CODED SIGNALS

by

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INTRODUCTION

In this study of cislunar and interplanetary electron density by radar techniques, it is frequently necessary to measure a group time delay or the differential of time delay on two or more frequencies. Though these measurements have been made successfully using simple sinusoidal modulation of a radio frequency carrier, it appears that important improvements can be achieved by the use of pseudorandom codes as the modulating waveform. These codes can be easily generated with linear feedback shift registers, and thus schemes employing them are practical to implement. The well known autocorrelation properties of pseudorandom codes have attracted many investigators and considerable success has already been achieved using these codes in space applications[1]. Most applications have made use of analog demodulators and signal processing.

In this paper we consider the problem of measuring differential time delay by the use of binary cross correlation of two received signals. Such an operation can be implemented with relative ease compared to analog correlation, especially if processing is to occur on a spacecraft. When both signals are very noisy, it requires approximately 2.46 as much time as analog processing; when one signal is clean, as would be the case in an absolute delay measurement, 2.02 as much time is needed.

Over

We assume both signals are corrupted by stationary, white, gaussian noise, bandlimited to the bit frequency of the pseudorandom code. This is done for mathematical simplicity based on digital sampling done once per bit. If the noise bandwidth is larger, and hence independent samples can be made more frequently, no change in any of the results will occur.

DISCUSSION

We recall that a pseudorandom sequence is a sequence of zeros and ones with the following well known properties:[2]

(1) It has length $M = 2^n - 1$, where n is a positive integer.

(2) It contains $\frac{M-1}{2}$ zeros and $\frac{M+1}{2}$ ones.

(3) It, together with its cyclic permutations and the all-zero sequence of length M, forms a group code under the operation of componentwise modulo two addition.

We will denote such a sequence as $S^{(0)}$ where

$$S^{(0)} = (s_1, s_2, \ldots s_M) \tag{1}$$

and shall denote the i^{th} cyclic permutation as

$$S^{(i)} = (s_{i+1}, s_{i+2}, \ldots . s_{i+M}) \tag{2}$$

where it is understood that all subscripts are taken modulo M, such that, for example

$$S_{j+kM} = s_j \, , \qquad k = \text{integer} \quad . \tag{3}$$

We note that each element of this group code is its own inverse.

The group property can be used to derive the autocorrelation of the pseudorandom sequence.

Given

$$S^{(k)} \quad \text{for} \quad k \neq 0, \quad \text{we have}$$

$$S^{(0)} + S^{(k)} = S^{(\ell)} \tag{4}$$

$$(\ell \neq k, \quad \ell \neq 0)$$

The sum of $S^{(0)}$ and $S^{(k)}$ has a one in every place in which $S^{(0)}$ and $S^{(k)}$ disagree and a zero in every place in which $S^{(0)}$ and $S^{(k)}$ agree. We know that $S^{(0)}$ has $\frac{M+1}{2}$ ones and $\frac{M-1}{2}$ zeros. Thus there is one more disagreement than agreement between $S^{(0)}$ and $S^{(k)}$; i.e.,

$$R(k) = -\frac{1}{M} \quad k \neq 0 \tag{5a}$$

where

$$R(k) = \frac{1}{M} \sum_{i=1}^{M} s_i \; s_{i+k} \tag{5b}$$

It is also clear that

$$R(0) = 1. \tag{5c}$$

Thus the sequence has autocorrelation as shown below:

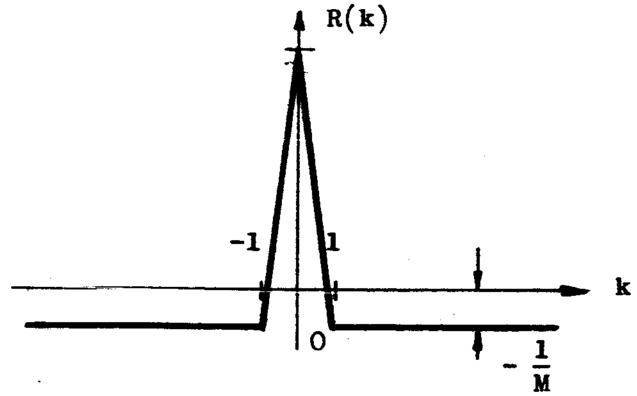

Fig. 1: AUTOCORRELATION OF PSEUDORANDOM SEQUENCE

We will assume that we have demodulated information received on two different carrier frequencies and thus must process two signals:

$$r_1(t) = A_1\, x(t) + n_1(t) \tag{6a}$$

$$r_2(t) = A_2\, x(t + \beta T) + n_2(t) \tag{6b}$$

Here $x(t)$ is a waveform switching between levels ± 1 in such a way that

$$x(t) = (-1)^{(s_i + 1)} \quad iT \leq t \leq (i + 1)T \tag{7}$$

i.e., is a pseudorandom waveform with bit time T.

We also assume that β is an integer with $0 \leq \beta < M$. The noises $n_1(t)$ and $n_2(t)$ are independent, stationary, zero-mean, gaussian, white, and are bandlimited to $(-\frac{1}{2T}, \frac{1}{2T})$ with power spectral densities of $\frac{N_1}{2}$ and $\frac{N_2}{2}$, respectively. Hence their autocorrelations are given by[3]

$$R_1(\tau) = \frac{N_1}{2T} \text{ sinc } \frac{\tau}{T} \tag{8a}$$

$$R_2(\tau) = \frac{N_2}{2T} \text{ sinc } \frac{\tau}{T} \tag{8b}$$

where

$$\text{sinc } x = \frac{\sin \pi x}{\pi x} \quad .$$

Our object is to obtain a reliable estimate of β. This will indicate differential group velocity of electromagnetic waves at the two carrier frequencies. Our estimate will be obtained with the system shown in Fig. 2. The signals $r_1(t)$ and $r_2(t)$ are hard limited, yielding their parities. Upon digital delay of $\text{sgn}\,[r_2(t)]$ and logical processing in which the two signals are compared once per bit, $\hat{R}(k)$ is obtained where we see that

$$\hat{R}(k) = \sum_{j=1}^{MN} \text{sgn}\,[r_1(t_o + jT)\, r_2(t_o + jT - kT)] \tag{9}$$

where processing starts at time t_o and ends at time $t_o + MNT$. We will define the estimated digital autocorrelation of $r_1(t)$ and $r_2(t)$ in a natural manner as

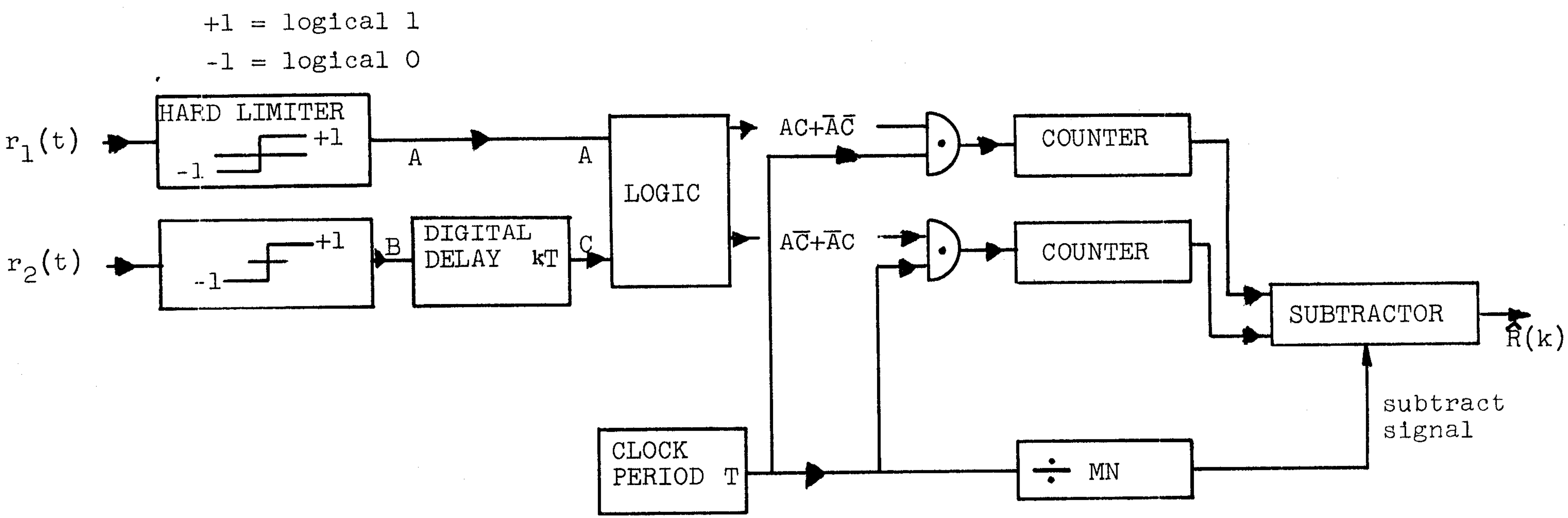

Fig. 2: Digital Processing of Received Signals

$$\hat{R}_N(k) = \frac{1}{MN} \hat{R}(k) \quad . \tag{10}$$

We should expect $\hat{R}_N(k)$ to have a maximum at $k = \beta$. Thus if $\hat{R}_N(k)$ is a consistent estimator, i.e., an estimate whose variance can be made arbitrarily small as the time for estimation is increased, we can determine β if we take N sufficiently large and vary k.

Let us define

$$p_1 = \text{probability that } n_1(t) > |A_1| \text{ at a fixed time } t' \tag{11a}$$

$$p_2 = \text{probability that } n_2(t) > |A_2| \text{ at a fixed time } t' \quad . \tag{11b}$$

We note that p_1 is just the probability that the parity of $r_1(t')$ is changed by noise; p_2 is just the probability that the parity of $r_2(t')$ is changed by noise. We now can easily calculate the statistics of our estimate, since we are sampling at times separated by integral multiples of T, at which the noise autocorrelations have zeros. Thus we only need use the stationary noise probability density functions. From eq. (10) we write

$$E[R_N(k)] = \frac{1}{MN} \sum_{j=1}^{MN} E\left\{\text{sgn}\left[r_1(t_o+jT)\ r_2(t_o+jT-kT)\right]\right\} \tag{12}$$

When $x(t_o+jT)$ and $x(t_o+jT-kT)$ have the same parity, the expected value of a sample is

$$Pr(\text{sample} = 1) - Pr(\text{sample} = -1) = (1-2p_1)(1-2p_2)\text{sgn}(A_1A_2) \tag{13a}$$

When $x(t_o+jT)$ and $x(t_o+jT-kT)$ have opposite parity, it is

$$\Pr(\text{sample} = 1) - \Pr(\text{sample} = -1) = -(1-2p_1)(1-2p_2)\,\text{sgn}(A_1A_2) \quad (13b)$$

Hence, from the autocorrelation of the sequence, we have

$$E[\hat{R}_N(\beta)] = (1-2p_1)(1-2p_2)\ \text{sgn}(A_1A_2) \quad (14a)$$

and for $k \neq \beta$

$$E[\hat{R}_N(k)] = -\frac{1}{M}\ (1-2p_1)(1-2p_2)\ \text{sgn}(A_1A_2) \quad . \quad (14b)$$

Thus the expected value of $\hat{R}_N(k)$ is merely the autocorrelation of the sequence scaled by the factor $(1-2p_1)(1-2p_2)\ \text{sgn}(A_1A_2)$.

Since samples taken are independent, we have

$$\begin{aligned} \text{var}[\hat{R}_N(k)] &= \left(\frac{1}{MN}\right)^2 \sum_{j=1}^{MN} \text{var}\left\{\text{sgn}[r_1(t_o+jT)\ r_2(t_o+jT-kT)]\right\} \\ &= \frac{1}{MN}\left[1 - (1-2p_1)^2\ (1-2p_2)^2\right] \end{aligned} \quad (15)$$

for any value of k. Hence our estimate is consistent.

We wish to compare the processing time required to that of an analog estimate using a matched filter in which the quantity of interest is

$$\hat{S}_N(k) = \frac{1}{MNT}\int_0^{MNT} r_1(t)\ r_2(t-kT)\ dt \ . \quad (16)$$

Such comparisons will be made for fixed ratios of mean to standard deviation. If we define

$$K_d = \frac{E[\hat{R}_N(k)]}{\left\{var[\hat{R}_N(k)]\right\}^{1/2}} \tag{17a}$$

$$K_a = \frac{E[\hat{S}_N(k)]}{\left\{var[\hat{S}_N(k)]\right\}^{1/2}} \tag{17b}$$

we find that:

1) If both signals have low signal-to-noise ratios, i.e., if

$$A_1^2 \ll \frac{N_1}{2T} \quad \text{and} \quad A_2^2 \ll \frac{N_2}{2T}, \quad \text{then} \quad \left(\frac{K_d}{K_a}\right)^2 = \frac{1}{2.46}$$

2) If one signal-to-noise ratio is low and there is no noise on the other channel, then

$$\left(\frac{K_d}{K_a}\right)^2 = \frac{1}{2.02} .$$

These results are derived in the appendix. We note that $\left(\frac{K_d}{K_a}\right)^2$ is the ratio of analog time to digital time, since means are time independent and variances are inversely proportional to processing time. Thus if both channels are noisy, 2.46 times as much processing time is required for our scheme. This must be balanced against relative simplicity of implementation. If one channel is clean, almost exactly twice as much time is required to go digital. For deep space probes where lightweight instrumentation is of great importance, the scheme clearly can be useful since increased processing time required will be unimportant in many practical situations.

APPENDIX

RATIO OF ANALOG TO DIGITAL PROCESSING TIMES

I. Two Noisy Channels

$$A_1^2 \ll \frac{N_1}{2T}, \quad A_2^2 \ll \frac{N_2}{2T}$$

In this case we have

$$(1 - 2p_1) \cong \sqrt{\frac{2}{\pi}} \frac{A_1}{\sigma_1} \tag{A-1a}$$

$$(1 - 2p_2) \cong \sqrt{\frac{2}{\pi}} \frac{A_2}{\sigma_2} \tag{A-1b}$$

where $\sigma_1^2 = \frac{N_1}{2T}$ and $\sigma_2^2 = \frac{N_2}{2T}$. Hence,

$$E[R_N(\beta)] = \frac{2}{\pi} \frac{A_1 A_2}{\sigma_1 \sigma_2} \tag{A-2}$$

Also

$$\mathrm{var}[\hat{R}_N(\beta)] = \frac{1}{MN} [1 - (1-2p_1)^2][1 - (1-2p_2)^2] \simeq \frac{1}{MN} \tag{A-3}$$

for low signal-to-noise ratios, since then $p_1 \approx p_2 \approx 1/2$.

Now consider the analog estimate.

$$E[\hat{S}_N(\beta)] = E[\frac{1}{MNT} \int_0^{MNT} r_1(t)\, r_2(t-\beta T)\, dt] = A_1 A_2 \tag{A-4}$$

Interchanging expectation and integration which is valid for bandlimited noise and finite integration time,

$$\operatorname{var}[\hat{S}_N(\beta)] = E\left\{\left[\frac{1}{MNT}\int_0^{MNT} r_1(t)\, r_2(t-\beta T)\, dt\right]^2\right\} - A_1^2 A_2^2$$

$$= E\left\{\frac{1}{M^2N^2T^2}\int_0^{MNT}\int_0^{MNT}[A_1^2 x(t)x(s)n_2(t)n_2(s)\right.$$

$$\left. + A_2^2 x(t)x(s)n_1(t)n_1(s) + n_1(t)n_1(s)n_2(t)n_2(s)]\, dtds\right\}$$

$$= \frac{1}{M^2N^2T^2}\int_0^{MNT}\int_0^{MNT}[A_1^2 < x(t)x(s) > R_2(t-s)$$

$$+ A_2^2 < x(t)x(s) > R_1(t-s) + R_1(t-s)\, R_2(t-s)]\, dtds \qquad \text{(A-5)}$$

Here brackets denote time averages, and terms with zero expectation have been dropped. Now, for $i = 1, 2,$ $< x(t)x(s) > R_i(t-s)$ is significant only for $|t-s| \leq T$. In this region we can write

$$< x(t)x(s) > \; R_i(t-s) = \sigma_i^2[1 - \frac{t-s}{T} \left(\frac{M+1}{M}\right) \operatorname{sinc} \frac{t-s}{T}] \; . \qquad \text{(A-6)}$$

We also assume that MNT is large enough such that

$$\int_0^{MNT} R_i(t-s)\, R_2(t-s)\, dt \simeq \int_{-\infty}^{\infty} R_1(t-s)\, R_2(t-s)\, dt \qquad \text{(A-7)}$$

for essentially all values of s under consideration. In this case

$$\operatorname{var}[\hat{S}_N(\beta)] = \frac{0.775}{MN}[A_1^2\sigma_2^2 + A_2^2\sigma_1^2] + \frac{\sigma_1^2\sigma_2^2}{MN}$$
$$\cong \frac{\sigma_1^2\sigma_2^2}{MN} \qquad \text{(A-8)}$$

using the fact that (4):

$$\int_{-1}^{1} \operatorname{sinc} x \;\; dx = 1.18 \qquad \text{(A-9)}$$

Thus

$$\left(\frac{K_d}{K_a}\right)^2 = \frac{1}{2.46} \qquad \text{(A-10)}$$

II. <u>One Noisy Channel and One Noiseless Channel</u>

$$A_1^2 \ll \frac{N_1}{2T}, \qquad N_2 = 0 .$$

Now we have

$$(1 - 2p_1) = \sqrt{\frac{2}{\pi}}\; \frac{A_1}{\sigma_1} \qquad \text{(A-11a)}$$

$$(1 - 2p_2) = 1 \qquad \text{(A-11b)}$$

Hence

$$E[R_N(\beta)] = \sqrt{\frac{2}{\pi}}\; \frac{A_1 A_2}{\sigma_1} \qquad \text{(A-12)}$$

Again

$$\mathrm{var}[R_N(\beta)] = \frac{1}{MN} \qquad \text{(A-13)}$$

For the analog estimate

$$E[S_N(\beta)] = A_1 A_2 \quad \text{as before} . \qquad \text{(A-14)}$$

Now, however,

$$\mathrm{var}[S_N(\beta)] = E \; \frac{1}{M^2 N^2 T^2} \int_0^{MNT} \int_0^{MNT} A_2^2 \, x(t)x(s)n_1(t)n_1(s)dtds$$

$$= \frac{0.775}{MN} A_2^2 \, \sigma_1^2 \quad \text{using eqn. (A-6)} \qquad \text{(A-15)}$$

Hence

$$\left(\frac{K_d}{K_a}\right)^2 = \frac{1}{2.02} \qquad \text{(A-16)}$$

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